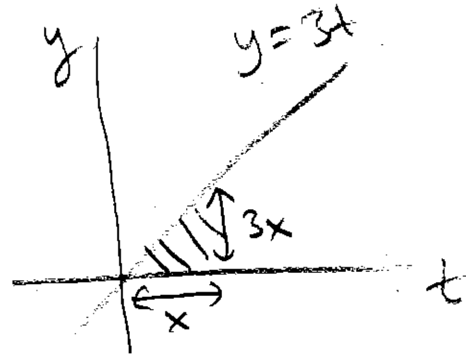
Closing Today: HW_1A,1B,1C
Closing next Wed: HW_2A,2B,2C
Read 5.3, 5.4 and 5.5 of the book.

5.3 The Fundamental Theorem of Calculus (FTOC)

Motivational Entry Task:
Consider the function *f(t) = 3t*.
Draw the graph and use area formulas you know, to compute:



1. $\int_0^1 f(t)dt = \frac{1}{2}(1)(3) = 3/2$

2. $\int_0^{10} f(t)dt = \frac{1}{2}(10)(30) = 150$

3. $\boxed{g(x) = \int_0^x f(t)dt = \frac{1}{2}(x)(3x) = \frac{3}{2}x^2}$

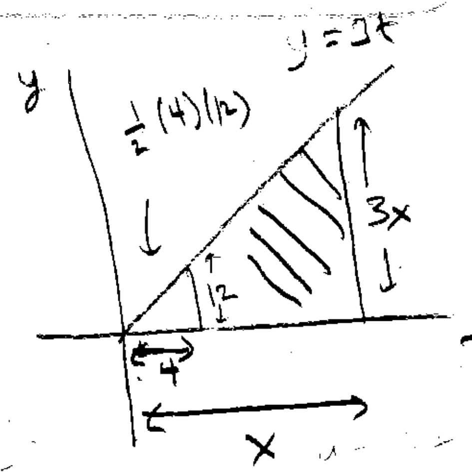
Any observations?

$\boxed{g'(x) = 3x}$

What changes if the lower bound a number other than 0? *For example*, let's try 4?

4. $h(x) = \int_4^x f(t)dt = \frac{3}{2}x^2 - 24$

$h'(x) = 3x$



Fundamental Theorem of Calculus (Part 1): *Areas under graphs are antiderivatives!*

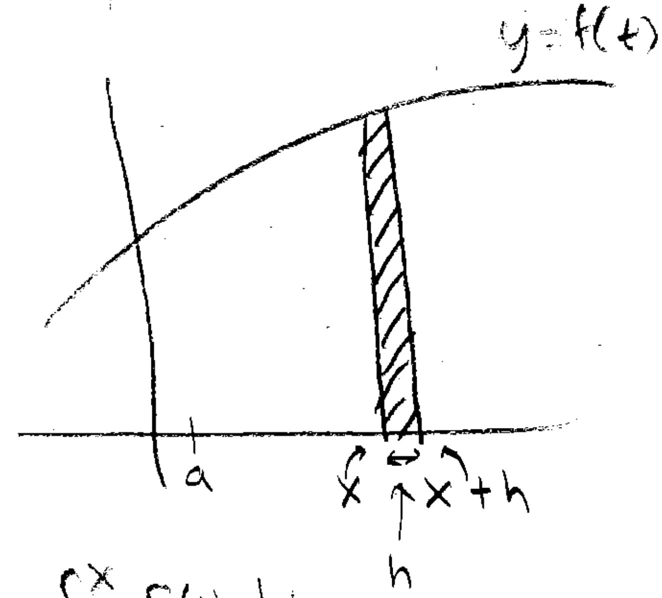
$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$$

That is, for any constant a, the "accumulated signed area" formula

$$F(x) = \int_a^x f(t)dt$$

is an antiderivative of $f(x)$.

ASIDE



$$g(x) = \int_a^x f(t)dt$$

$$g(x+h) - g(x) = \int_x^{x+h} f(t)dt$$

Thus, $$\frac{g(x+h)-g(x)}{h} = \frac{\int_x^{x+h} f(t)dt}{h}$$

$$\approx f(x)$$

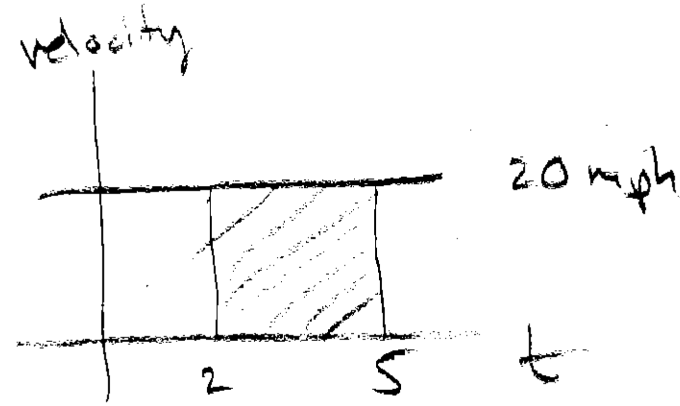
gets closer and closer as $h \to 0$

$$\underbrace{\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}}_{g'(x)} = f(x)$$

Motivation

Assume a car is traveling at a constant $v(t) = 20$ miles/hr.

What do the following represent?



1. $\displaystyle\int_2^5 \underbrace{v(t)}_{\frac{\text{miles}}{\text{hrs}}}\underbrace{dt}_{\text{hrs}} = 20(5-2) = 60$ miles

2. $\displaystyle f(x) = \int_2^x v(t)dt = 20(x-2) = 20x - 60$ MILES

DISTANCE FUNCTIONS!

3. $\displaystyle g(x) = \int_0^x v(t)dt = 20(x-0) = 20x$ MILES

Observations?

Mechanically using FTOC (Part 1)

Compute the **derivatives** of the following functions:

1. $g(x) = \int_{3}^{x} \cos(t)\,dt$

2. $h(x) = \int_{x}^{-2} te^t dt$

3. $f(x) = \int_{0}^{x^3} t + \sin(t)\,dt$

4. $k(x) = \int_{1+x^2}^{x^3} \sqrt{2+t}\,dt$

$\boxed{1}$ $g'(x) = \cos(x)$

FLIP

$\boxed{2}$ $h(x) = -\int_{-2}^{x} te^t dt$

$h'(x) = -xe^x$

$\boxed{3}$ CHAIN RULE!

$f'(x) = (x^3 + \sin(x^3))\, 3x^2$

$\boxed{4}$ SPLIT

$k(x) = \int_{1+x^2}^{0} \sqrt{2+t}\,dt + \int_0^{x^3} \sqrt{2+t}\,dt$

$= -\int_0^{1+x^2} \sqrt{2+t}\,dt + \int_0^{x^3} \sqrt{2+t}\,dt$

$k'(x) = -\sqrt{2+(1+x^2)}\,2x + \sqrt{2+x^3} \cdot 3x^2$

General form of FTOC (Part 1):

$$\boxed{\frac{d}{dx}\left(\int_{g(x)}^{h(x)} f(t)dt\right) = f(h(x))h'(x) - f(g(x))g'(x)}$$

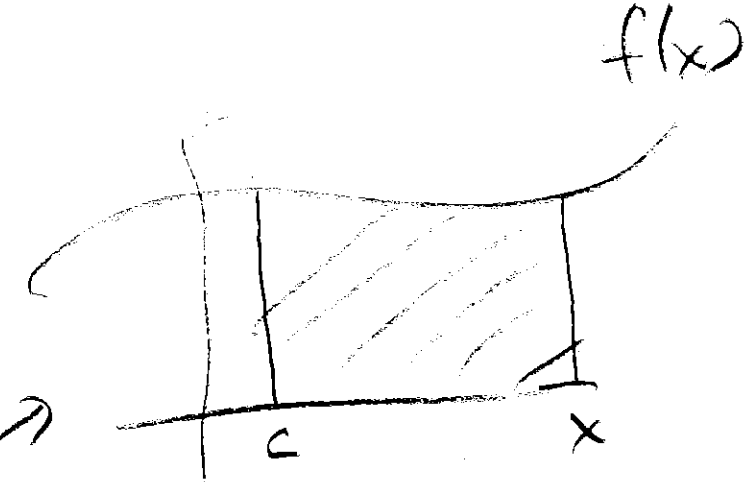
↳ IN OUR LAST EXAMPLE

$$\frac{d}{dx}\left(\int_{1+x^2}^{x^3} \sqrt{2+t}\,dt\right) = \sqrt{2+x^3}\,(3x^2) - \sqrt{2+(1+x^2)}\,(2x)$$

Fundamental Theorem of Calculus (Part 2):

If F(x) <u>any</u> antiderivative of f(x),

$$\int_a^b f(x)dx = F(b) - F(a)$$



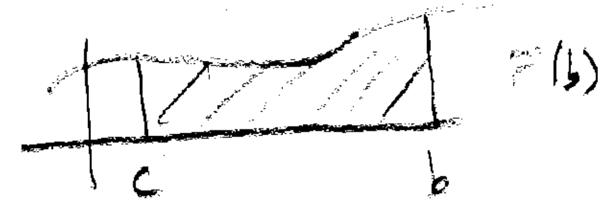
$\boxed{\text{ASIDE}}$

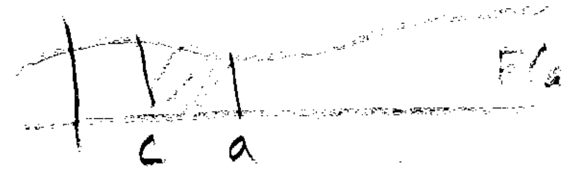
$\boxed{1}$ IF F(x) IS AN ANTIDERIVATIVE THEN IT REPRESENTS AN ACCUMULATED AREA FUNCTION FOR SOME STARTING VALUE c

$$F(x) = \int_c^x f(t)dt$$

$\boxed{2}$ SO F(b) = "AREA FROM c TO b"

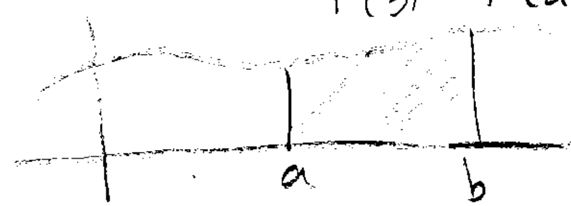
F(b) = "AREA FROM c TO a"





THUS, F(b) − F(a) = AREA BETWEEN a & b

F(b) − F(a)



<u>AND</u> IT DOESN'T MATTER WHAT THE VALUE OF c WAS!

Mechanically using FTOC (Part 2)

Evaluate

1. $\int_0^1 x^3\,dx = \frac{1}{4}x^4\Big|_0^1 = \frac{1}{4} - 0 = \boxed{1/4}$

$F(x) = \frac{1}{4}x^4 + C$

$F(b) - F(a)$

$F(1) - F(0)$

$\left[\frac{1}{4}(1)^4 + C\right] - \left[\frac{1}{4}(0)^4 + C\right]$

$\frac{1}{4} + \cancel{C} - \cancel{C}$

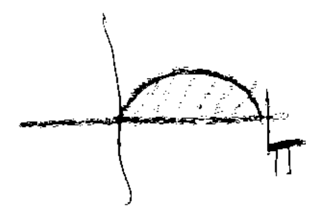
$\boxed{\frac{1}{4} = 0.25}$

2. $\int_0^{\pi} \sin(t)\,dt = -\cos(t)\Big|_0^{\pi}$

$= \left[-\cos(\pi)\right] - \left[-\cos(0)\right]$

$= -(-1) - -1$

$= \boxed{2}$



3. $\displaystyle\int_1^4 \frac{1}{w}\,dw$

$= \ln|w| \Big|_1^4$

$= \ln|4| - \ln|1|$

$= \boxed{\ln(4)}$

5. $\displaystyle\int_1^2 \frac{3}{x^2}\,dx = 3\int_1^2 x^{-2}\,dx$

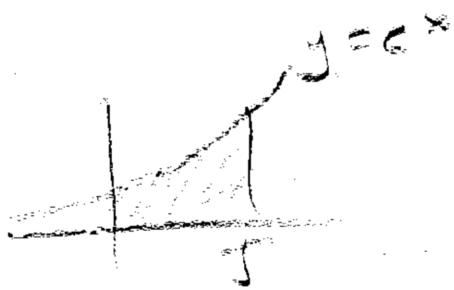
$= 3\left(\frac{1}{-1}x^{-1}\Big|_1^2\right)$

$= 3\left(-\frac{1}{x}\Big|_1^2\right)$

$= 3\left(\left[-\frac{1}{2}\right] - \left[-\frac{1}{1}\right]\right)$

$= \boxed{3/2}$

4. $\displaystyle\int_0^5 e^x\,dx$



$= e^x \Big|_0^5$

$= e^5 - e^0 = \boxed{e^5 - 1}$

6. $\displaystyle\int_1^4 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_1^4$

$= \frac{2}{3}\left(\left[4^{3/2}\right] - \left[1^{3/2}\right]\right)$

$= \frac{2}{3}(8-1) = \boxed{14/3}$

7. $\displaystyle\int_0^{1} \frac{1}{1+x^2}\,dx$

$= \tan^{-1}(x)\Big|_0^1$

$= [\tan^{-1}(1)] - [\tan^{-1}(0)]$

$= \pi/4 - 0 = \boxed{\pi/4}$

8. $\displaystyle\int_0^{\pi/3} \sec(x)\tan(x)\,dx$

$= \sec(x)\Big|_0^{\pi/3}$

$= \sec(\pi/3) - \sec(0)$

$= \dfrac{1}{\cos(\pi/3)} - \dfrac{1}{\cos(0)}$

$= \dfrac{1}{(1/2)} - \dfrac{1}{1}$

$= 2 - 1 = \boxed{1}$